Journal of Engineering Mathematics, Vol. 10, No. 1, January 1976
Noordhoff International Publishing–Leyden
Printed in The Netherlands

Flow with convective acceleration through a porous medium

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(Received January 22, 1975 and in revised form June 23, 1975)

SUMMARY
The flow streaming into a porous and permeable medium with arbitrary but smooth wall surface is considered on the basis of the Euler equation (in the pure fluid region) and a generalized Darcy's law in which the convective acceleration is taken into account. The asymptotic behavior of the flow for small permeability of the medium is investigated. It is shown that the flow in the porous medium is irrotational except in the boundary layer next to the surface. The velocity distribution in the boundary layer is given in a universal form. Proper boundary conditions connecting the potential flow in the pure fluid region and the potential flow in the porous medium are obtained when the boundary layer is neglected.

1. Introduction

It has become an important problem in many fields of engineering to remove small particles contained in gases, especially in air. One of the simple and useful methods of removal will be the one by means of a fibrous porous medium, i.e., a filter. Many studies on the filter have been done, e.g. the flow around fibers, the motion of a small particle in a fibrous medium, the efficiency of particle removal, the force acting on the fluid due to the medium, etc.. They concern the microscopic character of the flow through the filter. The book of Davies [1] may be referred to for these studies.

The investigation of the global flow through the filter is also important to understand the character of the filter and to make it more effective. For this study, Darcy's law may be used as a basic equation. This law expresses that the (sweepage) velocity is proportional to the pressure gradient and it does not have a convective acceleration of the fluid. This law is, therefore, considered to be valid for low speed flows, whereas the speed in the filter is not always small and the convective force may be important. To analyze this kind of flow, we should employ a generalized equation of Darcy's law in which the convection term is taken into account.*** Several studies [3–6] have been made on the basis of a generali-

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*** For a high speed flow, we may also pay attention to the force acting on the fluid by the porous medium. The force may deviate from the usual Darcy drag which is proportional to the velocity [2]. However, in the case of very porous media such as filters, the deviation will be small enough to be neglected (see Sec. 2).

zed Darcy's law accompanied by convection term. It is shown that vorticity emerges at the discontinuity surface of permeability when the fluid flows across the surface and that vorticity, if any, decays steadily in the flow where the permeability is constant. These interesting phenomena cannot be explained by the usual Darcy's law.

In a previous paper, Yamamoto and Yoshida [6] considered a suction and injection flow through a plane porous wall. They studied especially on the vortex layer attached to the surface in the wall and on the flow outside the vortex layer.

The present study is an extension and generalization of that work. That is, we consider a general suction flow into a porous medium with arbitrary but smooth surface. We treat the case of small permeability and investigate the asymptotic behavior of the flow for small parameter K which includes the permeability. In Sec. 2, we discuss the fundamental equation. We will get the solution in power series in K. It is shown in Sec. 3 that the main flow in the porous medium except in the boundary layer (vortex layer) next to the surface is generally irrotational. In Sec. 4, we investigate the boundary layer whose thickness is of $O(K)$ and obtain the velocity and pressure distributions in a universal form. Proper boundary conditions connecting the flow in the pure fluid (which is assumed to be irrotational) and the potential flow in the porous region are obtained when the boundary layer is neglected.

2. Fundamental equations

2.1. *General case for high porosity*

We consider a steady flow through an air filter, where the flow velocity is not always small. Most filters are made of fibrous materials and their porosity is very close to unity [1], [7]. In order to formulate the flow in such a very porous medium, we shall take a body force model: We regard the porous medium as an assemblage of small spherical particles fixed in space.* We take identical particles with radius σ and number density N. Since the porosity of the medium is close to unity, we take σ so that $\sigma^3 N \to 0$ while σN = finite when $N \to \infty$. Then the Reynolds number of a sphere $R_\sigma = \sigma V_*/\nu$ will be very small even when V_* is not small, where V_* is a reference speed and ν is the kinematic viscosity. The force on the sphere may be given by Stokes' formula, i.e. $F = 6\pi\sigma\mu V$, where V is the sweepage velocity and μ the viscosity of the fluid. The swarm of spheres exerts a force $6\pi\sigma\mu N V$ per unit volume on the fluid. Since σN = finite when $N \to \infty$, this body force is finite, while the porosity of this medium $(1 - \frac{4}{3}\pi\sigma^3 N)$ will tend to unity. Let a representative length of the macroscopic flow be L, which is in general much larger than σ. Then, the Reynolds number $R_L = V_* L/\nu$ in the present problem is not so small that we must take the convection term in the macroscopic equation of motion.

The viscous term due to the distortion of the velocity also should be taken into account for a general flow. Considering that the porosity of the medium is unity in the present body force model, i.e., the fluid occupies almost all parts of the porous medium, we may take that the viscous stress τ_{ij} is expressed in the same form as in a pure fluid:

$$\tau_{ij} = \mu\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right).$$

* The conception that the porous medium consists of spherical particles is widely used [7–11].

Then, with neglect of compressibility of the fluid, the fundamental equations in the porous medium are given by

$$\operatorname{div} V = 0, \tag{2.1}$$

$$\rho(V\cdot\operatorname{grad})V = -\operatorname{grad} P - \frac{\mu}{k} V + \mu\Delta V, \tag{2.2}$$

where P is the pressure, ρ the density, k is the permeability and is given by $(6\pi\sigma N)^{-1}$ for the body force model and Δ the Laplacian. We take V_*, ρV_*^2 and L as a reference speed, pressure and length, respectively. Then, eqs. (2.1) and (2.2) are rewritten in non-dimensional forms (we denote the non-dimensional variables by the same notations):

$$\operatorname{div} V = 0, \tag{2.1a}$$

$$(V\cdot\operatorname{grad})V = -\operatorname{grad} P - K^{-1}V + R_L^{-1}\Delta V, \tag{2.2a}$$

where

$$K \equiv (\rho V_* k)/(\mu L) = R_L(k/L^2). \tag{2.3}$$

When $R_L \ll 1$, the convection term (left-hand side) of eq. (2.2a) can be safely neglected and the equation is reduced to the one proposed by Brinkman [7], [9], [10]. Hence, it may be regarded as an extended equation of Brinkman's equation to a flow for $R_L \gtrsim 1$ through a very porous medium.

The Navier–Stokes equation is the most fundamental equation in a pure fluid region, that is,

$$\operatorname{div} \boldsymbol{u} = 0, \tag{2.4}$$

$$(\boldsymbol{u}\cdot\operatorname{grad})\boldsymbol{u} = -\operatorname{grad} p + R_L^{-1}\Delta \boldsymbol{u}, \tag{2.5}$$

where $\boldsymbol{u}$ and p are the non-dimensional velocity and pressure, respectively.

We next consider an appropriate boundary condition at the surface of the porous medium. We take a control volume straddling the surface and apply the conservation laws of mass and momentum. From the mass conservation law, we have

$$V_n = u_n \ (\equiv v_o), \tag{2.6}$$

where the subscript n means the normal component of the velocity to the surface. Taking into account that the force due to the porous medium is a body force*, we get the following equations from the momentum conservation law:

$$R_L^{-1}(\tau_t^+ - \tau_t^-) = v_o(u_t - V_t), \tag{2.7}$$

$$R_L^{-1}(\tau_n^+ - \tau_n^-) = p - P, \tag{2.8}$$

where the subscript t denotes the tangential components of the quantities to the surface and the signs + and − mean the values evaluated in the pure fluid region and in the porous medium, respectively. The tangential velocity in pores at the surface should be continuous

* The force at the porous surface exerted by the medium on the pure fluid may be $6\pi\sigma N/N^{\frac{1}{3}}$ per unit surface and hence this force will tend to zero as $N \to \infty$.

because of viscosity and we have one more condition:

$$V_t = u_t. \tag{2.9}$$

These are general fundamental equations and boundary conditions with which we can investigate a flow through a very porous medium. It will be interesting to study a general flow under the equations derived above. In the present paper, however, we shall mainly concern a flow of large Reynolds number ($R_L \gg 1$), because we are considering a high speed flow through a filter. The magnitude of the permeability of ordinary porous media is very small [7]. Regarding this, we consider a case $K \ll 1$ and investigate the asymptotic behavior of the flow for small K. We assume that the permeability k is constant. It is also assumed that the normal velocity v_o at the surface is $O(1)$.

2.2. *Fundamental equations for large Reynolds numbers*

Here, we discuss the equations and boundary conditions in case of large Reynolds numbers on the basis of the equations derived in Sec. 2.1. To begin with, we shall consider the flow in the porous medium. Since the Reynolds number R_L is very large, the viscous term in eqs. (2.2a) may be neglected except in a thin layer where the velocity changes abruptly. The fundamental equations are reduced to

$$\operatorname{div} V = 0, \tag{2.10}$$

$$(V \cdot \operatorname{grad})V = -\operatorname{grad} P - K^{-1}V. \tag{2.11}$$

A thin vortex layer will appear near the surface of the porous medium where the velocity changes very rapidly [6]. In this layer, the convection term is balanced with the body force term. We can deduce the thickness of the layer by simply comparing the order of both terms. Considering that the normal velocity v_o is order one, we will find this thickness to be $O(K)$. Then, the order of each term in eq. (2.2a) shows

$$(\text{Viscous})/(\text{Convection}) = (\text{Viscous})/(\text{Body force}) = (R_L K)^{-1}.$$

Hence, if we take

$$1 \gg K \gg R_L^{-1}, \tag{2.12}$$

the viscous term can be neglected even in a thin vortex layer. The viscous term has only a small effect of diffusing vorticity inside and outside the layer and the diffusion disappears as $R_L \to \infty$. Consequently, eq. (2.11) holds in a whole region of porous medium. A very porous medium may have the permeability of order of 10^{-4} cm^2. As an example, let us take $L = 10$ cm and $R_L = 10^4$, then we have $K = 10^{-2}$. In this case, the condition (2.12) is well satisfied. For a higher speed flow, R_L^{-1} becomes smaller, while K will be larger. It is therefore quite possible for real flows to satisfy the condition (2.12).

We take a similar consideration in the pure fluid region. The viscous term can be neglected for $R_L \gg 1$ in a region where the space derivative is order one and eqs. (2.4) and (2.5) become

$$\operatorname{div} \boldsymbol{u} = 0, \tag{2.13}$$

$$(\boldsymbol{u} \cdot \operatorname{grad})\boldsymbol{u} = -\operatorname{grad} p. \tag{2.14}$$

In a pure fluid region near the surface, there may be a viscous boundary layer which is a diffusion layer of vorticity produced at the surface. Considering that the flow has a normal velocity v_o at the surface and that the viscous force in eq. (2.5) is balanced with the convection term in the boundary layer, we will find that the thickness of the layer is of $O(R_L^{-1})$ and is very small compared with the vortex layer in the porous medium. The tangential derivative of the velocity in the vortex layer may be at most $O(K^{-1})$ [6] and so τ_t^- will be $O(K^{-1})$. Then, it will be seen from eqs. (2.7) and (2.9) that the variation of the velocity in the viscous layer is of $O((R_L K)^{-1})$ and is very small. The viscous boundary layer is not essential and can be neglected when the condition (2.12) is satisfied. We may treat the equation (2.14) in a whole pure fluid region.

As for the boundary condition, we cannot impose all boundary conditions (2.6)–(2.9) on the inviscid equations, because these consist of lower order derivatives. It may be consistent for the inviscid equation to take the boundary condition without considering the viscosity. We will get such boundary condition from the conservation laws of mass and momentum which should hold in this case, too. That is, by simply letting $R_L \to \infty$ in eqs. (2.6)–(2.8), we have [12]

$$\boldsymbol{u} = \boldsymbol{V}, \tag{2.15}$$

$$p = P. \tag{2.16}$$

The condition (2.15) says that there is no discontinuity of the velocity at the surface*. The velocity gradient and hence vorticity may, however, be discontinuous at the surface. If we take into account the viscosity, the vorticity diffuses by the action of the viscosity and a viscous boundary layer will appear near the surface. The variation of the velocity in the viscous boundary layer is, however, very small and tends to zero as $R_L \to \infty$. This means that the viscous boundary layer is not so important and can be neglected for large R_L.

It may be clarified from the above discussions that the inviscid equations without viscous terms and their corresponding boundary conditions instead of the original viscous equations can be treated in the analysis of a flow for $1 \gg K \gg R_L^{-1}$. To understand this more clearly, we consider a simple viscous flow on the basis of the viscous equations (2.1a), (2.2a), (2.4)–(2.9) and investigate the behavior of the flow when $1 \gg K \gg R_L^{-1}$.

We consider the same problem as studied in ref. [6], in which the inviscid equations are used. That is, we consider a constant suction flow into a semi-infinite plane porous wall. We take the y-axis normal to the wall directed towards the pure fluid region. The porous medium is in the region $y < 0$. We assume that the flow is independent of the x-coordinate lying on the wall surface. At $y \to \infty$, we may take $\boldsymbol{u} = (u, v) = (1, -v_\infty)$ and $p = 1$, where v_∞ is a positive constant. We assume that the flow does not diverge exponentially as $y \to -\infty$. It is easy to solve eqs. (2.1a), (2.2a), (2.4) and (2.5) together with the conditions (2.6)–(2.9) at $y = 0$ and the conditions at $y = \pm\infty$. We get the following results: in the pure fluid region ($y \geqq 0$)

$$u = 1 - \frac{\lambda}{R_L v_\infty + \lambda} \exp(-R_L v_\infty y), \qquad v = -v_\infty,\ p = 1,$$

* If there is no flow across the surface ($v_0 = 0$), the tangential velocity may be discontinuous. Then, the viscosity has an essential role to remove the discontinuity. This is not, however, the present case.

and in the porous medium ($y < 0$)

$$U = \frac{R_L v_\infty}{R_L v_\infty + \lambda} \exp(\lambda y), \quad V = -v_\infty, \; P = 1 + \frac{v_\infty}{K} y,$$

where

$$\lambda = -\tfrac{1}{2} R_L v_\infty + \sqrt{\frac{R_L^2 v_\infty^2}{4} + \frac{R_L}{K}}.$$

The vorticity produced at the surface diffuses to the pure fluid region owing to the viscosity and consequently a viscous boundary layer whose thickness is of $O(R_L^{-1})$ appears in the pure fluid region. If we take R_L in such a way that the condition (2.12) is satisfied, the above solution will reduce to

$$u = 1, \; v = -v_\infty, \; p = 1,$$

$$U = \exp\left(\frac{y}{K v_\infty}\right), \; V = -v_\infty, \; P = 1 + \frac{v_\infty}{K} y.$$

It will be seen that this flow is quite the same as is given in ref. [6]. This means that the analysis based on the inviscid equation (2.10), (2.11), (2.13)–(2.16) results in the same solution as is obtained from the viscous equation (2.1a), (2.2a), (2.4)–(2.9) when $1 \gg K \gg R_L^{-1}$.

Now, we proceed to investigate a high speed flow streaming into a porous body whose shape is smooth but arbitrary under the inviscid equations and boundary conditions. Equation (2.11) is written in another form:

$$V \times \operatorname{rot} V = \operatorname{grad}(P + \tfrac{1}{2} V \cdot V) + K^{-1} V. \tag{2.17}$$

Taking the rotation of the above equation, we get

$$(\boldsymbol{\Omega} \cdot \operatorname{grad}) V - (V \cdot \operatorname{grad}) \boldsymbol{\Omega} = K^{-1} \boldsymbol{\Omega}, \tag{2.18}$$

where

$$\boldsymbol{\Omega} = \operatorname{rot} V.$$

In this study, we consider the case where the flow in the pure fluid region is irrotational, i.e., the flow has a velocity potential ϕ

$$\boldsymbol{u} = \operatorname{grad} \phi. \tag{2.19}$$

Putting the above equation into eqs. (2.13) and (2.14), we get

$$\Delta \phi = 0, \tag{2.20}$$

$$p + \tfrac{1}{2} \boldsymbol{u} \cdot \boldsymbol{u} = \text{constant}. \tag{2.21}$$

3. Asymptotic field

First, we discuss the flow in the main region (asymptotic field) of the porous medium where the quantities of the flow do not change abruptly. Considering that K is a small quantity, we may expand the velocity and the pressure in power series in K:

$$\left.\begin{aligned} V &= V_D = V_D^{(0)} + KV_D^{(1)} + \dots, \\ P &= P_D = K^{-1}P_D^{(-1)} + P_D^{(0)} + KP_D^{(1)} + \dots . \end{aligned}\right\} \tag{3.1}$$

Here, it is assumed that the leading term of the pressure is of $O(K^{-1})$ to maintain the zeroth order flow. Inserting these expansions into eq. (2.18) and equating the same order terms in K, we can easily show

$$\Omega_D^{(i)} = \text{rot } V_D^{(i)} = 0 \quad (i = 0, 1, 2, \dots). \tag{3.2}$$

That is, the flow in the asymptotic field is irrotational in any order in K. The flow has a velocity potential Φ_D and

$$V_D^{(i)} = \text{grad } \Phi_D^{(i)} \quad (i = 0, 1, 2, \dots), \tag{3.3}$$

where

$$\Phi_D = \Phi_D^{(0)} + K\Phi_D^{(1)} + \dots . \tag{3.3a}$$

Putting eqs. (3.1)–(3.3) into eq. (2.17), we get

$$P_D + \tfrac{1}{2}V_D \cdot V_D + K^{-1}\Phi_D = \text{const.} \tag{3.4}$$

It is easily shown that the first terms of eq. (3.1) ($V_D^{(0)}$, $P_D^{(-1)}$) satisfy the usual Darcy's law, i.e.,

$$V_D^{(0)} = -\text{grad } P_D^{(-1)}, \; \Delta P_D^{(-1)} = 0. \tag{3.5}$$

That is, the main character of the flow in the asymptotic field is determined by the pressure gradient only and the effect of convection exists in higher order terms. The pressure gradient along the wall surface of the porous medium is generally order one. Then, eq. (3.5) shows that the tangential velocity near the surface in the asymptotic field is at most $O(K)$. On the other hand, the tangential velocity at the surface is assumed to be $O(1)$. There is a big velocity difference between two tangential velocities, and the boundary condition (2.15) is not satisfied. There must be a boundary layer near the surface where the convection affects the flow directly. The fluid streaming into the porous medium losses a large amount of tangential momentum in a thin layer near the surface. The flow has vorticity in the layer [6]. We next consider this vortex layer.

4. Vortex layer

4.1. *Fundamental equations in orthogonal curvilinear coordinates*

It is convenient for the analysis of the vortex layer (boundary layer) adjacent to the surface to introduce orthogonal curvilinear coordinates. We take the x_3 as a coordinate along the unit normal $\boldsymbol{n}$ to the boundary (pointed into the porous medium) and the x_1, x_2 coordinates within the parallel surface x_3 = const., then an arbitrary position vector is expressed by

$$\boldsymbol{x} = x_3\boldsymbol{n}(x_1, x_2) + \boldsymbol{x}_w(x_1, x_2), \tag{4.1}$$

where $\boldsymbol{x}_w = (x_w, y_w, z_w)$ is the position vector on the surface.

For simplicity, we shall take the coordinates x_1 and x_2 in such a way that the coordinate lines coincide with the lines of principal curvature of the boundary. Then, the system of coordinates is triply orthogonal everywhere. Let R_1 and R_2 be the principal radii of curvature of the boundary, where R_i is the radius in the direction of x_i and is normalized by L. We take $R_i > 0$ when the normal points to the center of principal curvature. From eq. (4.1) and Rodrigues' formula [13], the metrical coefficients [14] are given by

$$\frac{1}{H_i} = \left(1 - \frac{x_3}{R_i}\right)\frac{1}{A_i} \quad (i = 1, 2),\ H_3 = 1, \tag{4.2}^\star$$

where

$$\frac{1}{A_i^2} = \left(\frac{\partial x_w}{\partial x_i}\right)^2 + \left(\frac{\partial y_w}{\partial x_i}\right)^2 + \left(\frac{\partial z_w}{\partial x_i}\right)^2. \tag{4.2a}$$

The fundamental equations (2.10) and (2.17) are now rewritten as follows:

$$H_1 H_2 \left\{\frac{\partial}{\partial x_1}\left(\frac{V_1}{H_2}\right) + \frac{\partial}{\partial x_2}\left(\frac{V_2}{H_1}\right) + \frac{\partial}{\partial x_3}\left(\frac{V_3}{H_1 H_2}\right)\right\} = 0, \tag{4.3}$$

$$(-1)^{i+1} H_1 H_2 \frac{V_1 V_2}{V_i}\left\{\frac{\partial}{\partial x_1}\left(\frac{V_2}{H_2}\right) - \frac{\partial}{\partial x_2}\left(\frac{V_1}{H_1}\right)\right\} - H_i V_3 \left\{\frac{\partial}{\partial x_3}\left(\frac{V_i}{H_i}\right) - \frac{\partial V_3}{\partial x_i}\right\}$$
$$= H_i \frac{\partial}{\partial x_i}\{P + \tfrac{1}{2}(V_1^2 + V_2^2 + V_3^2)\} + K^{-1} V_i \quad (i = 1, 2), \tag{4.4a, b}$$

$$H_1 V_1 \left\{\frac{\partial}{\partial x_3}\left(\frac{V_1}{H_1}\right) - \frac{\partial V_3}{\partial x_1}\right\} - H_2 V_2 \left\{\frac{\partial V_3}{\partial x_2} - \frac{\partial}{\partial x_3}\left(\frac{V_2}{H_2}\right)\right\}$$
$$= \frac{\partial}{\partial x_3}\{P + \tfrac{1}{2}(V_1^2 + V_2^2 + V_3^2)\} + K^{-1} V_3, \tag{4.5}$$

where V_i is the x_i-component of the velocity. The boundary conditions at $x_3 = 0$ for these equations are

$$u_i = V_i, \tag{4.6}$$

$$p = P. \tag{4.7}$$

4.2. *Analysis of the vortex layer*

We now proceed to the analysis of the vortex layer. In this layer, the tangential momentum of the fluid entering into the porous medium across the surface decreases owing to the force exerted by the porous medium. Considering that $V_i \sim O(1)$ at the surface and that the convection term in eq. (4.4) is balanced with the body force term, we find that the thickness of the layer is of $O(K)$.

$\star$ Here and below, the double suffix does not mean the usual dummy suffix. For example, $A_i A_i$ is $A_1 A_1$ or $A_2 A_2$, etc.

Then, we shall introduce a new stretched coordinate related to x_3 by

$$x_3 = K\eta. \tag{4.8}$$

Taking into account that the flow will be irrotational as $\eta \to \infty$, we may put the solution in the vortex layer in the following forms:

$$\left.\begin{aligned} V &= V_D(\boldsymbol{x}; K) + V_B(x_1, x_2, \eta; K), \\ P &= P_D(\boldsymbol{x}; K) + P_B(x_1, x_2, \eta; K). \end{aligned}\right\} \tag{4.9}$$

The correction terms V_B and P_B should vanish as $\eta \to \infty$. We expand the correction terms in power series in K:

$$\left.\begin{aligned} V_B &= V_B^{(0)} + KV_B^{(1)} + \ldots, \\ P_B &= K^{-1}P_B^{(-1)} + P_B^{(0)} + KP_B^{(1)} + \ldots. \end{aligned}\right\} \tag{4.10}$$

We should rewrite the asymptotic quantities (3.1) in the new variable. Expanding these quantities in power series in K, we get the reordening of V_D and P_D, i.e.,

$$\left.\begin{aligned} V_D &= V_D^{(0)}(0) + K\left\{\left(\frac{\partial V_D^{(0)}}{\partial x_3}\right)_0 \eta + V_D^{(1)}(0)\right\} \\ &\quad + K^2\left\{\frac{1}{2}\left(\frac{\partial^2 V_D^{(0)}}{\partial x_3^2}\right)_0 \eta^2 + \left(\frac{\partial V_D^{(1)}}{\partial x_3}\right)_0 \eta + V_D^{(2)}(0)\right\} + \ldots, \\ P_D &= K^{-1}P_D^{(-1)}(0) + \left\{\left(\frac{\partial P_D^{(-1)}}{\partial x_3}\right)_0 \eta + P_D^{(0)}(0)\right\} \\ &\quad + K\left\{\frac{1}{2}\left(\frac{\partial^2 P_D^{(-1)}}{\partial x_3^2}\right)_0 \eta^2 + \left(\frac{\partial P_D^{(0)}}{\partial x_3}\right)_0 \eta + P_D^{(1)}(0)\right\} + \ldots, \end{aligned}\right\} \tag{4.11}$$

where the notations $F(0)$ and $(F)_0$ mean the values evaluated at $x_3 = 0$. The solution in the pure fluid region is also expanded in power series in K:

$$\left.\begin{aligned} \boldsymbol{u} &= \boldsymbol{u}^{(0)} + K\boldsymbol{u}^{(1)} + K^2\boldsymbol{u}^{(2)} + \ldots, \\ p &= p^{(0)} + Kp^{(1)} + K^2p^{(2)} + \ldots, \end{aligned}\right\} \tag{4.12}$$

where

$$\operatorname{rot} \boldsymbol{u}^{(i)} = 0 \quad (i = 0, 1, 2, \ldots). \tag{4.12a}$$

Then, the boundary conditions (4.6) and (4.7) at $\eta = 0$ are rewritten as

$$u_j^{(i)} = V_{D,j}^{(i)} + V_{B,j}^{(i)} \quad (i = 0, 1, 2, \ldots), (j = 1, 2, 3), \tag{4.13a, b, c}$$

$$0 = P_D^{(-1)} + P_B^{(-1)}, \tag{4.14a}$$

$$p^{(i)} = P_D^{(i)} + P_B^{(i)} \quad (i = 0, 1, 2, \ldots). \tag{4.14b}$$

We also have the following conditions as $\eta \to \infty$

$$V_{B,j}^{(i)} \to 0, \; P_B^{(i-1)} \to 0 \quad (i = 0, 1, 2, \ldots). \tag{4.15}$$

We substitute eqs. (4.2) and (4.9) together with the expansions (4.10) and (4.11) into the fundamental equations (4.3)–(4.5). Expanding the results in power series in K and equating the same order terms, we get the equations for the correction terms of the boundary layer. From the K^{-1}th order term of eq. (4.3), we have

$$\partial V_{B,3}^{(0)}/\partial\eta = 0.$$

Considering the boundary condition (4.15), we get the following solution

$$V_{B,3}^{(0)} = 0. \tag{4.16}$$

Therefore, the boundary condition (4.13c) becomes

$$u_3^{(0)}(0) = V_{D,3}^{(0)}(0) \;\; (> 0). \tag{4.17}$$

Putting eq. (4.16) into eq. (4.5) and considering the condition (4.15), we find that the K^{-1}th and K^0th order corrections of the pressure are zero; i.e.,

$$P_B^{(-1)} = 0, \;\; P_B^{(0)} = 0. \tag{4.18a, b}$$

From eqs. (4.14a) and (4.18a), we have

$$P_D^{(-1)}(0) = 0. \tag{4.19}$$

Equation (3.5) then leads to

$$V_{D,j}^{(0)}(0) = 0 \qquad (j = 1, 2). \tag{4.20}$$

We next obtain the velocity distribution in the vortex layer. Using the above results (4.16)–(4.18) in eq. (4.4), we have the following equations for $V_{B,j}^{(0)}$

$$u_3^{(0)}(0)\frac{\partial V_{B,j}^{(0)}}{\partial\eta} + V_{B,j}^{(0)} = 0 \qquad (j = 1, 2). \tag{4.21a, b}$$

Solutions of these equations subject to the boundary condition (4.13) together with (4.20) are given by

$$V_{B,j}^{(0)} = u_j^{(0)}(0)\exp\{-\eta/u_3^{(0)}(0)\} \qquad (j = 1, 2). \tag{4.22a, b}$$

It is found that the flow has vorticity. In the pure fluid region, we take a potential flow, while vorticity emerges suddenly when the flow crosses the surface. This production of vorticity is due to the tangential force to the surface exerted by the porous medium and essentially due to the viscosity of the fluid which appears in the form of a body force.

We proceed to the next order approximation. Inserting eqs. (4.16)–(4.20) into eqs. (4.3)–(4.5), we get the following first order equations

$$\frac{\partial V_{B,3}^{(1)}}{\partial\eta} = -A_1A_2\left\{\frac{\partial}{\partial x_1}\left(\frac{V_{B,1}^{(0)}}{A_2}\right) + \frac{\partial}{\partial x_2}\left(\frac{V_{B,2}^{(0)}}{A_1}\right)\right\}, \tag{4.23}$$

$$u_3^{(0)}(0)\frac{\partial V_{B,j}^{(1)}}{\partial\eta} + V_{B,j}^{(1)} = (-1)^{j+1}\frac{V_{B,1}^{(0)}V_{B,2}^{(0)}}{V_{B,j}^{(0)}}$$

$$\times\left\{\frac{\partial}{\partial x_1}\left(\frac{V_{B,2}^{(0)}}{A_2}\right) - \frac{\partial}{\partial x_2}\left(\frac{V_{B,1}^{(0)}}{A_1}\right)\right\}(A_1A_2) - \tfrac{1}{2}A_j\frac{\partial}{\partial x_j}\{(V_{B,1}^{(0)})^2 + (V_{B,2}^{(0)})^2\}$$

$$+ u_3^{(0)}(0)\frac{V_{B,j}^{(0)}}{R_j} - \left\{V_{D,3}^{(1)}(0) + \left(\frac{\partial V_{D,3}^{(0)}}{\partial x_3}\right)_0\eta + V_{B,3}^{(1)}\right\}\left(\frac{\partial V_{B,j}^{(0)}}{\partial\eta}\right) \quad (j = 1, 2), \qquad (4.24\text{a, b})$$

$$\frac{\partial P_B^{(1)}}{\partial\eta} = -\left\{R_1^{-1}(V_{B,1}^{(0)})^2 + R_2^{-1}(V_{B,2}^{(0)})^2 + A_1\left(\frac{\partial u_3^{(0)}}{\partial x_1}\right)_0 V_{B,1}^{(0)}\right.$$

$$\left. + A_2\left(\frac{\partial u_3^{(0)}}{\partial x_2}\right)_0 V_{B,2}^{(0)} + u_3^{(0)}(0)\frac{\partial V_{B,3}^{(1)}}{\partial\eta} + V_{B,3}^{(1)}\right\}. \qquad (4.25)$$

We solve these equations under the conditions of (4.13) and (4.15). The calculation is very tedious but straightforward and the final results are given by

$$V_{B,3}^{(1)} = A_1A_2\sum_{l=1}^{2}\frac{\partial}{\partial x_l}\left\{\frac{A_l}{A_1A_2}u_l^{(0)}(0)u_3^{(0)}(0)\exp(-\eta/u_3^{(0)}(0))\right\}, \qquad (4.26)$$

$$V_{B,j}^{(1)} = \left[u_j^{(1)}(0) - V_{D,j}^{(1)} + \frac{u_j^{(0)}(0)}{u_3^{(0)}(0)}\right.$$

$$\left.\times\left\{\left(\frac{u_3^{(0)}(0)}{R_j} + \frac{V_{D,3}^{(1)}(0)}{u_3^{(0)}(0)}\right)\eta + \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\eta^2\right\}\right]\exp(-\eta/u_3^{(0)}(0))$$

$$+\left[\frac{u_j^{(0)}(0)}{u_3^{(0)}(0)}V_{B,3}^{(1)}(0) - \frac{A_j}{2}\frac{\partial}{\partial x_j}\{(u_1^{(0)}(0))^2 + (u_2^{(0)}(0))^2\}\right]$$

$$\times[\exp(-\eta/u_3^{(0)}(0)) - \exp(-2\eta/u_3^{(0)}(0))], \qquad (4.27\text{a, b})$$

$$P_B^{(1)} = u_3^{(0)}(0)\sum_{l=1}^{2}\left\{\frac{1}{2R_l}(u_l^{(0)}(0))^2\exp(-2\eta/u_3^{(0)}(0))\right.$$

$$\left. + 2A_lu_l^{(0)}(0)\left(\frac{\partial u_3^{(0)}}{\partial x_l}\right)_0\exp(-\eta/u_3^{(0)}(0))\right\}. \qquad (4.28)$$

On the way of the calculation, we used the relation (4.12a) concerning the potential flow of the outer region. Inserting eqs. (4.22) and (4.27) into the second order continuity equation (cf. (4.3)) and solving the resulting equation with (4.15), we get the normal velocity $V_{B,3}^{(0)}$ of the second order as:

$$V_{B,3}^{(2)} = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\eta V_{B,3}^{(1)}$$

$$+ A_1A_2\sum_{l=1}^{2}\frac{\partial}{\partial x_l}\left[\frac{A_l}{A_1A_2}u_3^{(0)}(0)\left\{\left[u_l^{(1)}(0) - V_{D,l}^{(1)}(0) + \right.\right.\right.$$

$$+ \frac{u_l^{(0)}(0)}{u_3^{(0)}(0)} \left\{ \left(2 \frac{u_3^{(0)}(0)}{R_l} + \frac{V_{D,3}^{(1)}}{u_3^{(0)}(0)} \right) (\eta + u_3^{(0)}(0)) + \frac{1}{2} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \eta^2 \right\} \Bigg]$$

$$\times \exp(-\eta/u_3^{(0)}(0))$$

$$+ \left[\frac{u_l^{(0)}(0)}{u_3^{(0)}(0)} V_{B,3}^{(1)}(0) - \frac{A_l}{2} \frac{\partial}{\partial x_l} \{(u_1^{(0)}(0))^2 + (u_2^{(0)}(0))^2\} \right]$$

$$\times [\exp(-\eta/u_3^{(0)}(0)) - \tfrac{1}{2} \exp(-2\eta/u_3^{(0)}(0))] \Bigg\} \Bigg]. \tag{4.29}$$

Thus, the solution in the vortex layer is given in a universal form.

4.3. *The boundary condition at the wall surface for both potential flows*

Inserting the solution (4.18), (4.26), (4.28) and (4.29) into the other boundary conditions (4.13c) and (4.14) which have not been used so far, we get the following equations at $x_3 = 0$:

$$P_D^{(-1)}(0) = 0, \tag{4.30}$$

$$u_3^{(0)}(0) = V_{D,3}^{(0)}(0), \tag{4.31a}$$

$$p^{(0)}(0) = P_D^{(0)}(0), \tag{4.31b}$$

$$u_3^{(1)}(0) = V_{D,3}^{(1)}(0) + A_1 A_2 \sum_{l=1}^{2} \frac{\partial}{\partial x_l} \left(\frac{A_l}{A_1 A_2} u_l^{(0)} u_3^{(0)} \right)_0, \tag{4.32a}$$

$$p^{(1)}(0) = P_D^{(1)}(0) + u_3^{(0)}(0) \sum_{l=1}^{2} \left\{ \frac{1}{2R_l} (u_l^{(0)}(0))^2 + 2A_l \left(\frac{\partial u_3^{(0)}}{\partial x_l} \right)_0 u_l^{(0)}(0) \right\}, \tag{4.32b}$$

$$u_3^{(2)}(0) = V_{D,3}^{(2)}(0) + A_1 A_2 \sum_{l=1}^{2} \frac{\partial}{\partial x_l} \left[\frac{A_l}{A_1 A_2} u_3^{(0)}(0) \left\{ u_l^{(1)}(0) - V_{D,l}^{(1)}(0) \right. \right.$$

$$+ u_l^{(0)}(0) \left(2 \frac{u_3^{(0)}(0)}{R_l} + \frac{1}{2} \frac{u_3^{(1)}(0) + V_{D,3}^{(1)}(0)}{u_3^{(0)}(0)} \right)$$

$$\left. \left. - \frac{A_l}{4} \frac{\partial}{\partial x_l} [(u_1^{(0)}(0))^2 + (u_2^{(0)}0))^2] \right\} \right]. \tag{4.33}$$

These equations are the relationships to be satisfied at the surface by both potential flows of the pure fluid and porous regions and thus constitute the boundary conditions for these potential flow. That is, under these boundary conditions, we may solve the following Laplace equation,

the pure fluid region

$$\left. \begin{aligned} &\Delta \phi^{(i)} = 0; \ \Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2, \\ &\boldsymbol{u}^{(i)} = \operatorname{grad} \phi^{(i)} \quad (i = 0, 1, 2, \ldots), \end{aligned} \right\} \tag{4.34}$$

the pressure is given by the Bernoulli equation:

$$p^{(0)} + \tfrac{1}{2}\boldsymbol{u}^{(0)}\cdot\boldsymbol{u}^{(0)} = (\text{const.})_0, \tag{4.35a}$$

$$p^{(1)} + \boldsymbol{u}^{(0)}\cdot\boldsymbol{u}^{(1)} = (\text{const.})_1, \tag{4.35b}$$

$$\cdots\cdots\cdots\cdots$$

the asymptotic field of the porous region

$$\left.\begin{aligned} &\Delta\Phi_D^{(i)} = 0 \qquad (i = 0, 1, 2, \ldots), \\ &V_D^{(i)} = \text{grad}\,\Phi_D^{(i)}, \end{aligned}\right\} \tag{4.36}$$

the pressure is given by (cf. (3.4))

$$P_D^{(-1)} + \Phi_D^{(0)} = (\text{Const.})_{-1}, \tag{4.37a}$$

$$P_D^{(0)} + \Phi_D^{(1)} + \tfrac{1}{2}V_D^{(0)}\cdot V_D^{(0)} = (\text{Const.})_0, \tag{4.37b}$$

$$P_D^{(1)} + \Phi_D^{(2)} + V_D^{(0)}\cdot V_D^{(1)} = (\text{Const.})_1, \tag{4.37c}$$

$$\cdots\cdots\cdots\cdots$$

It is fairly easy to solve the Laplace equation, because it is a linear equation. After solving the potential flows, we can get the boundary layer solution by use of the formulas in Sec. 4.2.

Acknowledgements

The authors express their cordial thanks to Professor Ko Tamada at Kyoto University for interest and valuable discussion. Part of this work was done while the senior author (K.Y.) was staying at Physics Department of Eindhoven University of Technology as a Research Fellow. He would like to express his cordial thanks to Professor G. Vossers for his hospitality and encouragement and he is also deeply indebted to Mr. M. E. H. van Dongen for valuable discussion.

The discussion in Sec. 2 which concerns the effect of viscosity originated from a referee's comment. The authors owe much to him in the point that the fundamental equations may be well understood with this discussion.

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